Supplemental Material for "Swirling fluid reduces the bounce of partially filled containers"

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(Dated: May 4, 2023)

Consider a cylindrical bottle of radius r_2, height h, and mass m_b partially filled with a mass m_l of liquid of density ρ. The bottle is set in rotation at angular velocity $\omega = 2\pi\Omega$ before its release, so the liquid inside also rotates steadily (Ω is the rotational frequency). After release, the cylinder falls freely under the acceleration gravity, reaching a solid flat surface at $t = 0$ with velocity $-v$. Before the collision, we assume that the liquid inside the container has reached its new steady state, *i.e.* the centrifugal force has pushed away all the liquid onto the walls, forming a cylindrical shell of height h (the full extent of the bottle), outer radius r_2 and inner radius r_1. The elastic hemisphere attached to the cylinder bottom has a radius R. The equation of motion for the cylinder is

$$m_b \ddot{z} = -m_b g + F_e - F_d - F_l, \tag{1}$$

where F_e is the elastic force due to the compression of the hemisphere, $F_d = \gamma \dot{z}|z|\Theta(-z)$ is a nonlinear damping force due to dissipative processes during contact and deformation of the hemisphere [1], and F_l is an interaction force between the liquid and the container. We consider a contact force law between the elastic hemisphere and the impact surface given by

$$F_e = \frac{4}{3}\frac{E}{1-\nu^2} R^{1/2} |z|^2 \Theta(-z), \tag{2}$$

where E is the elastic modulus, ν the Poisson ratio, and Θ is the Heaviside function, which accounts for situations when the hemisphere is not in contact with the solid ground.

A SIMPLIFIED MODEL FOR THE ELASTIC COLLISION

Let us assume that after the collision, a portion of the shell of water with mass $\rho\pi\left(r_2^2 - r_1^2\right) v_- \Delta t$, which is on the wall, is focused after the collision into a section of a jet of radius r_j upward velocity v_j and constant angular velocity ω_j. Mass conservation law during the collision leads to

$$r_j^2 (v_j - \dot{z}) - \left(r_2^2 - r_1^2\right)(v_- + \dot{z}) = 0. \tag{3}$$

Likewise, the conservation of angular momentum implies that

$$\omega_j r_j^4 (v_j - \dot{z}) - \omega\left(r_2^4 - r_1^4\right)(v_- + \dot{z}) = 0. \tag{4}$$

The change of momentum due to the action of the force exerted on the container F_l yields

$$F_l = \frac{\Delta p}{\Delta t} = \rho\pi\left[r_j^2 (v_j - \dot{z}) v_j + \left(r_2^2 - r_1^2\right)(v_- + \dot{z}) v_-\right]. \tag{5}$$

Energy conservation under the action of the force F_l acting on the bottom of the bottle, which travels at speed $\dot{z}$, requires that the power exerted on the fluid equals the rate of change of total kinetic energy of the hitting fluid and the emerging jet. This can be written as

$$F_l \cdot \dot{z} = \frac{1}{2}\rho\pi\left\{r_j^2\left(v_j^2 + \frac{1}{2}\omega_j^2 r_j^2\right)(v_j - \dot{z}) - \left(r_2^2 - r_1^2\right)\left[v_-^2 + \frac{1}{2}\omega^2\left(r_2^2 + r_1^2\right)\right](v_- + \dot{z})\right\}. \tag{6}$$

The system of equations (3), (4), (5), and (6) allows obtaining r_j, v_j, ω_j, and F_l in terms of $\dot{z}$ and the parameters of the system ρ, r_1, r_2, v_-, and ω. From equations (3) and (4), we can obtain the following dimensionless coefficients,

$$\alpha_\pm \equiv \frac{r_2^2 \pm r_1^2}{r_j^2}, \quad \beta \equiv \frac{\alpha_+}{\alpha_-} = \frac{2}{\phi} - 1, \quad \phi \equiv 1 - \frac{r_1^2}{r_2^2},$$

where we have introduced ϕ as the fraction of volume occupied by the fluid. From these expressions, we note that

$$\frac{v_j - \dot{z}}{v_- + \dot{z}} = \alpha_-, \quad \frac{v_j}{v_-} = \alpha_- + (1 + \alpha_-)\frac{\dot{z}}{v_-}, \quad \frac{\omega_j}{\omega} = \alpha_+.$$

Dividing equation (6) by (5) to eliminate F_l, one obtains

$$\left(1 - \alpha_-^2\right)\left(1 + \frac{\dot{z}}{v_-}\right)^2 = \xi\left(\beta\alpha_- - 1\right), \tag{7}$$

where we have defined $\xi \equiv E_r/E_t$ as the ratio between the rotational and translational kinetic energies before impact, with $E_r \equiv \pi\rho\omega^2\left(r_2^4 - r_1^4\right)/4$ and $E_t \equiv \pi\rho v_-^2\left(r_2^2 - r_1^2\right)/2$, which corresponds to the rotational and translational energy, respectively. Equation (7) leads to

$$\alpha_- = -\frac{1}{2}\epsilon\beta + \sqrt{\left(\frac{1}{2}\epsilon\beta\right)^2 + \epsilon + 1}, \tag{8}$$

where $\epsilon \equiv \xi\left(1 + \dot{z}/v_-\right)^{-2}$. Thus, the force in Eq. (5) is

$$F_l = \pi\rho\left(r_2^2 - r_1^2\right)v_-^2\left(\sqrt{\left(\frac{1}{\phi}\xi\right)^2(1-\phi) + \left[\left(1 + \frac{\dot{z}}{v_-}\right)^2 + \frac{1}{2}\xi\right]^2} - \frac{\xi}{\phi} + \left(1 + \frac{\dot{z}}{v_-}\right)^2 + \frac{\xi}{2}\right)\Theta\left(v_- t + h - z\right), \tag{9}$$

where the Heaviside step function considers the constraint that the force F_l can be only exerted by the remaining liquid available on the wall.

DIMENSIONLESS FORMULATION

To understand the role of each parameter of the model and reduce the total number of parameters, it is convenient to write the equation of motion in dimensionless form. Setting the following scaling laws: $z \sim h$, $t \sim h/v_-$, and consistently, $\dot{z} \sim v_-$, $\ddot{z} \sim v_-^2/h$; plus $F_e \sim (4/3)R^3E/(1-\nu^2)$, $F_d \sim \gamma$, and $F_l \sim \pi\rho r_2^2 v_-^2$; Eq. (1) can be written as

$$\ddot{\bar{z}} = -\Pi_g + \Pi_e f_e(\bar{z}) - \Pi_d f_d(\bar{z}, \dot{\bar{z}}) - \Pi_l f_l(\bar{z}, \dot{,\bar{z}}), \tag{10}$$

where

$$\Pi_g \equiv \frac{gh}{v_-^2}, \quad \Pi_e \equiv \frac{4}{3}\left(\frac{1}{1-\nu^2}\right)\left(\frac{h}{R}\right)^{5/2}\frac{R^3E}{m_b v_-^2}, \quad \Pi_d \equiv \frac{\gamma h}{m_b v_-^2}, \quad \Pi_l \equiv \frac{\rho V_b}{m_b}, \quad \Pi_\Omega \equiv \frac{\omega^2 r_2^2}{v_-^2}.$$

Parameter ξ in terms of the dimensionless parameters is $\xi = \Pi_\Omega\left(1 - \phi/2\right)$, and the dimensionless forces are

$$f_e\left(\bar{z}\right) \equiv \left|\bar{z}\right|^2\Theta\left(-\bar{z}\right), \tag{11}$$

$$f_d\left(\bar{z}, \dot{\bar{z}}\right) \equiv \dot{\bar{z}}\left|\bar{z}\right|\Theta\left(-\bar{z}\right), \tag{12}$$

$$f_l\left(\bar{z}, \dot{\bar{z}}\right) \equiv \phi\left[\sqrt{\left(\frac{\xi}{\phi}\right)^2(1-\phi) + \left(\left[1 + \dot{\bar{z}}\right]^2 + \frac{\xi}{2}\right)^2} - \left(\frac{\xi}{\phi}\right) + \left(\left[1 + \dot{\bar{z}}\right]^2 + \frac{\xi}{2}\right)\right]\Theta\left(1 - \bar{z} - \bar{t}\right) \tag{13}$$

with $\bar{z}$ and $\bar{t}$ the dimensionless z-coordinate and time, respectively.

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